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Laminar Flow in a Channel with a Step

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SUMMARY
In this paper we investigated numerically the laminar flow of a homogeneous, viscous incompressible liquid, through a channel with a step. We used one numerical approach for small Reynolds numbers, and a different method for large numbers. Streamlines were calculated with high accuracy using a relatively small amount of computer time.

1. Introduction

The flow of a two-dimensional, steady-state, viscous incompressible liquid through a channel with a step, was investigated in detail by D. Greenspan [1]. Boundary conditions consisted of an initially parabolic velocity profile, and a horizontal flow and constant pressure downstream.

Our study concentrated upon the same problem. For small Reynolds numbers R, we preferred a different approach than was developed in [2]. For moderate and high Reynolds numbers we proceeded with Greenspan's original numerical scheme but modified the computational stage. This helped to reduce the computer time needed for convergence by a factor of 10. We have also dealt with one of the problem's boundary conditions—namely the constant pressure downstream—in a different manner than [1]. Thus we extended the range of convergence from small R to all Reynolds numbers. Finally we compiled a table of computer times for comparison.

2. Formulation of the Problem

Denote by u, v the horizontal and normal velocity component measured in terms of an average initial velocity u_0; p is the pressure in terms of ρu_0^2, where ρ is the fluid density; x, y are the horizontal and normal coordinate in units of a—the width of the channel. The Navier–Stokes equations are then

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{R}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{R}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.2}$$

where

$$R = \frac{au_0}{\nu} \tag{2.3}$$

is the Reynolds number.

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 . \tag{2.4}$$

Equation (2.4) introduces a stream function $\psi(x, y)$ for which

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} .$$

Define the vorticity $\zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

and get a new set of equations in ψ, ζ:

$$\Delta\psi = -\zeta \tag{2.6}$$

$$\Delta\zeta + R\left(\frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x}\right) = 0\,. \tag{2.7}$$

The channel with a step is shown by the polygon ABCDEFGH in Figure 1.

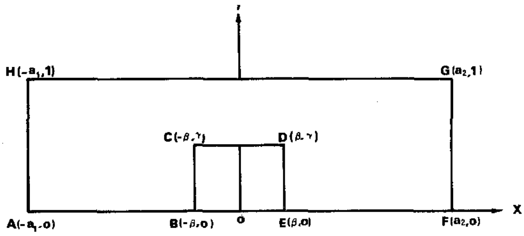

Figure 1.

The boundary conditions are

$$\psi = 1,\ \frac{\partial\psi}{\partial y} = 0, \quad \text{on HG} \tag{2.8}$$

$$\psi = 0,\ \frac{\partial\psi}{\partial y} = 0, \ \text{on AB, DC, EF} \tag{2.9}$$

$$\psi = 0,\ \frac{\partial\psi}{\partial x} = 0, \quad \text{on BC, DE} \tag{2.10}$$

$$\psi = 3y^2 - 2y^3,\ \zeta = 12y - 6, \quad \text{on AH} \tag{2.11}$$

$$\frac{\partial\psi}{\partial x} = 0,\ \frac{\partial\zeta}{\partial x} - R\frac{\partial\psi}{\partial y}\left(\zeta + \frac{\partial^2\psi}{\partial y^2}\right) = 0, \quad \text{on FG}\,. \tag{2.12}$$

Equation (2.11) assumes an initial parabolic velocity profile, while (2.12) satisfies the requirements of horizontal flow and constant pressure far downstream.

Remark: In [1] the second condition of (2.12) appeared as

$$\frac{\partial\zeta}{\partial x} + R\frac{\partial\psi}{\partial y}\left(\zeta + \frac{\partial^2\psi}{\partial y^2}\right) = 0\,,$$

and as a result a wrong finite difference was deduced. However, a formal correction of the sign would have led towards a diverging procedure for moderate and large Reynolds numbers, due to a possible vanishing denominator in the difference equation. In order to solve this problem we used an indirect approach for the boundary FG, and ended with (3.10).

3. The Numerical Method

We solved (2.6)–(2.7) by relaxation and distinguished between two different cases:

a. *Small Reynolds Numbers*

We followed the technique suggested in [2], [3] for a flow in a circular pipe.

Let o be an internal grid point, with neighbouring points 1, 2, 3, 4. Denote by h the size of the mesh (Figure 2). We replaced (2.6), (2.7) respectively by the difference equations:

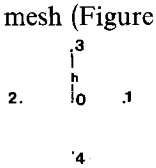

Figure 2.

$$\psi_o = \frac{\psi_1+\psi_2+\psi_3+\psi_4+h^2\zeta_o}{4} \equiv L_1(\psi,\zeta)\,. \tag{3.1}$$

$$\zeta_o = \frac{\zeta_1+\zeta_2+\zeta_3+\zeta_4+\frac{1}{4}R[(\psi_1-\psi_2)(\zeta_3-\zeta_4)-(\psi_3-\psi_4)(\zeta_1-\zeta_2)]}{4} \equiv L_2(\psi,\zeta) \tag{3.2}$$

and used them to relax ψ_o, ζ_o in the following manner:

$$\zeta_o^{(n+1)} = \omega L_2\ (\psi_i^{(n^*)}, \zeta_i^{(n^*)}\ ,\ i=1,\dots,4)+(1-\omega)\zeta_o^{(n)} \tag{3.3}$$

$$\psi_o^{(n+1)} = \omega L_1\ (\zeta_o^{(n+1)}\ ,\ \psi_i^{(n^*)},\ i=1,\dots,4)+(1-\omega)\psi_o^{(n)} \tag{3.4}$$

In (3.3), (3.4) ω is an overrelaxation factor ; $\psi_o^{(n)}$, $\zeta_o^{(n)}$ are the current and $\psi_o^{(n+1)}$, $\zeta_o^{(n+1)}$ the iterated values of ψ, ζ at the $(n+1)$th iteration; $n^*=n+1$ if the corresponding $\psi_i^{(n+1)}$, $\zeta_i^{(n+1)}$ was already computed in the course of the sweep; otherwise we took $n^*=n$.

At boundary points other than AH, the interior of FG, and B, C, D, E, we used the formula:

$$\zeta_o = \frac{3(\psi_o-\psi_1)}{h^2} - \frac{\zeta_1}{2} \equiv L_3(\psi,\zeta) \tag{3.5}$$

where o is a boundary point and 1 a neighbouring internal point in a normal direction.

Following [1] we derived at C, D:

$$\zeta_C = -\frac{1}{h^2}[\psi(-\beta,\gamma+h)+\psi(-\beta-h,\gamma)] \equiv L_4(\psi) \tag{3.6}$$

$$\zeta_D = -\frac{1}{h^2}[\psi(\ \ \beta,\gamma+h)+\psi(\ \ \beta+h,\gamma)] \equiv L_5(\psi)\,. \tag{3.7}$$

Let o be internal to FG with neighbouring points 1, 2, 3, 4, 5 (Figure 3).

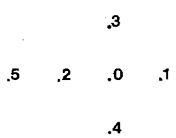

Figure 3.

Since 1 is outside the mesh, ψ_1, ζ_1 should not appear in the final formulae for ψ_o, ζ_o. However we may use them in the calculation.

First we replaced the second part of (2.12) by

$$\frac{\zeta_1-\zeta_2}{2h} = R\,\frac{\psi_3-\psi_4}{2h}\left(\zeta_o+\frac{\psi_3+\psi_4-2\psi_o}{h^2}\right) \tag{3.8}$$

and then since (3.2) holds, eliminated ζ_1 in terms of ψ_1, ζ_o, and $(\psi_i, \zeta_i, i=2,3,4)$:

$$\zeta_1 = \frac{\zeta_2+\zeta_3+\zeta_4-4\zeta_o+\dfrac{R}{4}[(\psi_1-\psi_2)(\zeta_3-\zeta_4)+\zeta_2(\psi_3-\psi_4)]}{\dfrac{R}{4}(\psi_3-\psi_4)-1}\,. \tag{3.9}$$

We substituted this in (3.8) and finally got

$$\zeta_o = \frac{2\zeta_2+\zeta_3+\zeta_4+\dfrac{R}{4}(\psi_1-\psi_2)(\zeta_3-\zeta_4)-M\left(\dfrac{M}{4}-1\right)\dfrac{\psi_3+\psi_4-2\psi_o}{h^2}}{4+M\left(\dfrac{M}{4}-1\right)} \equiv L_6(\psi,\zeta) \tag{3.10}$$

where $M=R(\psi_3-\psi_4)$ and $\psi_1=\psi_2$.

The first condition of (2.12) was replaced by

$$\psi_o = \frac{4\psi_2 - \psi_5}{3} \equiv L_7(\psi) \tag{3.11}$$

We used (3.5), (3.6), (3.7), (3.10), (3.11) to relax ψ_o, ζ_o by

$$\zeta_o^{(n+1)} = \omega_B L_i + (1-\omega_B)\zeta_o^{(n)}, \qquad i=3, \ldots, 6 \tag{3.12}$$

$$\psi_o^{(n+1)} = \omega_B L_7 + (1-\omega_B)\psi_o^{(n)} \tag{3.13}$$

where ω_B is a different overrelaxation factor.

b. *Moderate and Large Reynolds Numbers*

In (3.2) R appears in the numerator only, and therefore we could not expect convergence for large Reynolds numbers. The technique developed in [1], practically a linearization of (2.6)–(2.7), leads to a convergent solution for all R. It introduces two finite sequences of discrete functions

$$\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(k)}, \psi^{(k+1)} \tag{3.14}$$

$$\zeta^{(0)}, \zeta^{(1)}, \ldots, \zeta^{(k)}, \zeta^{(k+1)}. \tag{3.15}$$

If for some tolerance ε

$$|\psi^{(k)} - \psi^{(k+1)}| < \varepsilon \tag{3.16}$$

$$|\zeta^{(k)} - \zeta^{(k+1)}| < \varepsilon \tag{3.17}$$

at each point of the mesh, $\psi^{(k)}$, $\zeta^{(k)}$ are taken as approximations to ψ, ζ respectively.

However, in order to compute each $\psi^{(i)}$ or $\zeta^{(i)}$, a complete relaxation procedure is needed. In terms of computer time, the method is inefficient, and unless one is satisfied with a rough mesh of $h=\frac{1}{10}$, he may need many hours of computer time for each problem.

We modified this technique by using direct relaxation like in case (a).

Equation (3.1) remained the same. Determine the values

$$H = \psi_1 - \psi_2, \quad K = \psi_3 - \psi_4. \tag{3.18}$$

If for example $H \geqslant 0$, $K \geqslant 0$, ζ_x, ζ_y in (2.7) were replaced by

$$\zeta_x = \frac{\zeta_o - \zeta_2}{h} \tag{3.19}$$

$$\zeta_y = \frac{\zeta_3 - \zeta_o}{h} \tag{3.20}$$

and (3.2) by:

$$\zeta_o = \frac{\zeta_1+\zeta_2+\zeta_3+\zeta_4 + \frac{R}{2}(\zeta_3 H + \zeta_2 K)}{4 + \frac{R}{2}(H+K)} \equiv L_8(\psi, \zeta). \tag{3.21}$$

The other three possibilities are described in detail in [1].

At boundary points other than those internal to FG the same formulae as in [1] had been used. At boundary points internal to FG (3.10) held and (3.11) has been replaced by the simple relation

$$\psi_o = \psi_2. \tag{3.22}$$

During one complete iteration we followed [1] and used 1 as an overrelaxation factor for ζ at all grid points. We denoted the result by $\bar{\zeta}^{(n+1)}$ and finally took

$$\zeta^{(n+1)} = 0.3\bar{\zeta}^{(n+1)} + 0.7\zeta^{(n)}. \tag{3.23}$$

The overrelaxation factor for ψ was taken as 1.8 for all grid points, and we finally had ([1]):

$$\psi^{(n+1)} = 0.96\bar{\psi}^{(n+1)} + 0.04\psi^{(n)} \tag{3.24}$$

This technique overcomes the instability of equation (3.2) by introducing R in both the numerator and the denominator. Obviously we loose some accuracy, but the procedure leads to convergence for all R.

4. Numerical Results

For computations we used the CDC 3600, thus being able to compare our results to those obtained in [1].

We also chose $\beta=1$, $\gamma=\frac{1}{2}$, $h=\frac{1}{10}, \frac{1}{20}, \frac{1}{40}$ and compiled Table 1 for purposes of comparison.

We calculated streamlines and vorticities for $R=0$, 1, 10, 20, 50, 100, 200, 500, 1000 and sketched streamlines for $R=0$, 50, 500 and $h=\frac{1}{40}$ (Figures 4, 5, 6).

One finds immediately that the running time used in [1] has been reduced by a factor of 10. This fact enabled us to solve the problem for a smaller h ($\frac{1}{20}, \frac{1}{40}$) within a reasonable amount of computer time. We got highly accurate results, and also discovered an important effect that had been missing in [1]—a vortex near the corner B $(-\beta, 0)$.

TABLE 1

R	a_1	a_2	h	Our approximate running time (min)	running time in [1] (min)
10	4	4	0.1	0.5	6
50	4	4	0.1	0.3	8
100	4	4	0.1	0.6	10
200	4	4	0.1	1.1	12
500	4	10	0.1	5	50
1000	4	10	0.1	10	91

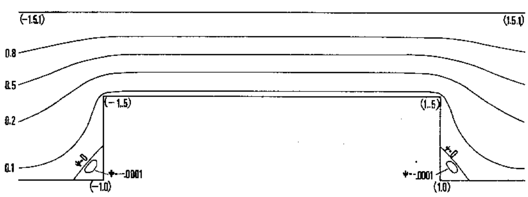

Figure 4. Streamlines for $R=0$.

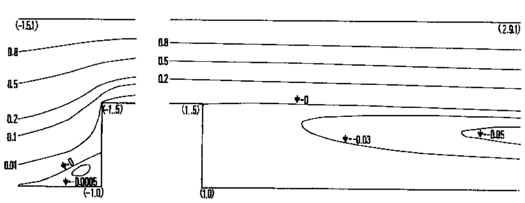

Figure 5. Streamlines for $R=50$.

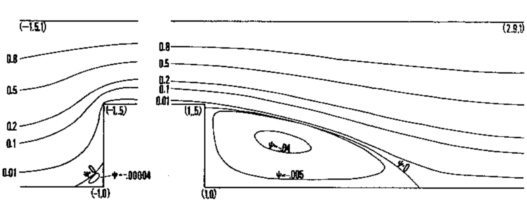

Figure 6. Streamlines for $R=500$.

The vortex exists for $R=0$ (Figure 4), then reduces (or diminishes completely) to some minimum for a R_{min}: $10 < R_{\text{min}} < 50$ (Figure 5), and finally increases monotonically with R (Figure 6). In [1] the author used only $h=\frac{1}{10}$, and could not possibly discover this vortex for $R < 200$.

The reduction in computer time may of course be increased. This depends on the ability to find optimal overrelaxation factors. However, this matter is another problem, and is under study.

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